Algebraic Geometry: MIDTERM SOLUTIONS

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Abstract. Algebraic Geometry: MIDTERM. We give terse solutions to this Midterm Exam.

1. Problem 1:

Problem 1 (Geometry 1).

(1) *Let $I_1, I_2 \subset R$ be ideals. Then show that $rad(I_1 I_2) = rad(I_1 \cap I_2)$.*
(2) *Let $I \subset R$ be an ideal. Suppose there exists a prime ideal P such that $I^k \subset P \subset I$ for some $k \in \mathbb{Z}^+$. Show that $p = I = rad(I)$.*

Solution 1.

(1) *A prime ideal contains $I_1 I_2$ if and only if the prime ideal contains $I_1 \cap I_2$.*
(2) *Take radicals.*

2. Problem 2:

Problem 2 (Geometry 2). *Find all the automorphisms of $\mathbb{A}^1$.*

Solution 2.

(1) *If k is algebracally closed then any regular funtion $f : \mathbb{A}^1 \longrightarrow k$ is a polynomial in one variable.*
(2) *If k is not algebraically closed then a regular function f need not be a polynomial function. For example for the field of real numbers $f : \mathbb{R} \longrightarrow \mathbb{R}$ we can take $f(x) = \frac{1}{x^2+1}$.*
(3) *If k is an infinite field like an algebraically closed field then the space of polynomial functions and polynomials agree.*
(4) *If f is injective polynomial then $f(x) = a$ has to have at most one root for all $a \in k$.*
(5) *If k is algebraically closed of characteristic 0 then f must be linear with $f(x) = ax + b$ with $a \neq 0, a, b \in k$. Also f^{-1} is a linear polynomial and a regular function.*
(6) *If k is not algebraically closed of characteristic 0 say the field of real numbers then consider the degree three polynomial $f(x) = x^3$. It is injective but not linear.*

(7) *If k is algebraically closed of characteristic p where p is a prime then $f(x) = g(x^p)$ where g is an injective polynomial and hence by induction on degree*

$$f(x) = ax^{p^n} + b.$$

(8) *If f, f^{-1} are both polynomials of degree n, m respectively then*

$$f \circ f^{-1} = 1_k = f^{-1} \circ f.$$

So by a degree argument we have f and f^{-1} both must be linear.

(9) *If k is algebraically closed then $f(x) = ax + b$ and f^{-1} is also linear and a regular function.*

(10) *For an algebraically closed field*

$$Aut(\mathbb{A}^1) = \{T_{(a,b)} : k \longrightarrow k \mid T(x) = ax + b, a \in k^*, b \in k\}.$$

(11) *Some more facts:*

- *If*
  - *k is a finite field*
  - *or k is an algebraically closed field of characteristic zero*

  then any bijective regular function $\psi : k \longrightarrow k$ is given by a polynomial in one variable which also has a regular polynomial inverse.
- *This is not true for an algebracally closed field of characteristic p.*
- *This is not true for a non-algebracally closed field of characteristic* 0.

3. Problem 3

PROBLEM 3 (Geometry 3). *Let I_1, I_2 be ideals in the polynomial ring $k[x_1, x_2, \ldots, x_n]$. Show that for if I_2 is not contained in any of the associated primes of I_1 then $(I_1 : I_2) = I_1$.*

SOLUTION 3. *Let $I_1 = \bigcap_{i=1}^{r} \mathcal{Q}_i$ be its primary decomposition with $rad(\mathcal{Q}_i) = \mathcal{P}_i$. Let $x \in (I_1 : I_2)$ then $xI_2 \subset I_1$. Let $y_i \in I_2 \backslash \mathcal{P}_i$. Then $xy_i \in \mathcal{Q}_i \Rightarrow x \in \mathcal{Q}_i$. So $x \in \bigcap_{i=1}^{r} \mathcal{Q}_i = I_1 \Rightarrow (I_1 : I_2) \subset I_1$. Also clearly $I_1 \subset (I_1 : I_2)$. So the problem follows.*

4. Problem 4

PROBLEM 4 (Geometry 4). *Let $I \subset R$ be an ideal and $a \in R$. Suppose there exists for some integer $M \geq 0$ we have $(I : a^M) = (I : a^{M+1})$. Show that $\bigcup_m (I : a^m) = (I : a^M)$.*

SOLUTION 4. *We observe that $(I : a^j) \subset (I : a^i)$ for all $i \geq j$. We have for any $b \in R$*

$$ba^M \in I \Longleftrightarrow ba^{M+1} \in I.$$

So we also have $ba^{M+2} \in I$ then $baa^{M+1} \in I$. So $baa^M = ba^{M+1} \in I \Rightarrow ba^M \in I$. So by induction we have for $j \geq M$ if $ba^j \in I$ then $ba^M \in I$. So the problem follows.

5. Problem 5

PROBLEM 5 (Geometry 5). *Let R be a commutative ring with unity. Consider the set $Spec(R) = \{p \in R \mid p \text{ is a prime ideal}\}$. Describe $\mathbb{C}[X], \mathbb{R}[X]$.*

SOLUTION 5. *We observe that every polynomial over $\mathbb{C}[X]$ factorizes completely and every odd degree polynomial in $\mathbb{R}[x]$ has a root.*

(1) *The prime ideals in $\mathbb{C}[X]$ are (0) and $(x-a)$ where $a \in \mathbb{C}$.*
(2) *The prime ideals in $\mathbb{R}[X]$ are (0) and $(x^2+ax+b), (x-c)$ where $c \in \mathbb{R}$ and x^2+ax+b is a quadratic irreducible.*

6. Problem 6

PROBLEM 6 (Geometry 6). *Describe all rational maps $\mathbb{A}^1 \longrightarrow C$ where $C \subset \mathbb{A}^2$ is the curve given by $V(y^2-x^3)$.*

SOLUTION 6. *The space $\mathbb{A}^1$ is irreducible hence any open subset of $\mathbb{A}^1$ is dense and irreducible. Hence the image of an open set under a regular map is also irreducible. Now C has a cofinite topology. So the irreducible subsets are*

(1) *A single point.*
(2) *Any infinite subset of $\mathbb{A}^1$ (cofinite topology) is also irreducible topologically.*

Now consider this particular polynomial map $\psi : \mathbb{A}^1 \longrightarrow C$ given by $t \longrightarrow (t^2, t^3)$ with a rational inverse defined on $C \backslash \{(0,0)\}$ given by $(x,y) \longrightarrow \frac{y}{x}$. Using this map if $\phi : \mathbb{A}^1 \longrightarrow C$ is rational map then $\psi^{-1} \circ \phi : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ is rational map and hence must be a rational function. So there exist $p \in \mathbb{C}[t], q \in C[t]\backslash\{0\}$ such that

$$\psi^{-1}(\phi(t)) = \frac{p(t)}{q(t)} \Rightarrow \phi(t) = \big((\frac{p(t)}{q(t)})^2, (\frac{p(t)}{q(t)})^3\big).$$

So the space of all rational maps is described as

$$HomRat(\mathbb{A}^1, C) = \{\phi_{p,q} \mid p \in \mathbb{C}[t], q \in \mathbb{C}[t]\backslash\{0\}, \phi(t) = \big((\frac{p(t)}{q(t)})^2, (\frac{p(t)}{q(t)})^3\big)\}.$$

Actually the image of rational map $\phi_{p,q}$ is either a single point or misses the point $(0,0)$. The domain of the rational map is $\{t \in \mathbb{A}^1 \mid q(t) \neq 0\}$ a complement of a finite set in $\mathbb{A}^1$.

7. Problem 7

PROBLEM 7 (Geometry 7). *Don't assume k is algebraically closed for this problem. Let $\mathcal{A} = (xy, yz, zx) \subset k[x,y,z]$. Is $\mathcal{A} = I(V(\mathcal{A}))$? Prove that $\mathcal{A}$ cannot be generated by 2 elements.*

SOLUTION 7.

(1) *Let $f \in k[x,y,z]$. We observe the following that f^n has monomial terms in single variables x, y, z alone or a constant term if and only if f^n has monomial terms in single variables x, y, z alone or a constant term. With this observation we immediately conclude that $\mathcal{A}$ is a radical ideal.*
(2) *Now for all fields k, $V(\mathcal{A}) = (X-axis) \cup (Y-axis) \cup (Z-axis)$.*

(3) *If k is a non-algebraically closed field then let $p \in k[x]$ be a non-constant polynomial in one variable which has no roots in k. Then its homogenization $P[x,y] = y^{deg(p)}p(\frac{x}{y})$ has only origin $(0,0)$ as a root. This must have single monomial terms in both the variables x, y. Now consider the polynomial $F[x,y,z] = P[P[x,y],z]$. This polynomial also must have single monomial terms in each of the variables x,y,z. Now the function F vanishes only at the origin in k^3.*

(4) *Let $k = \mathbb{F}_q$ a finite field. Now the roots of the polynomial*

$$G[x,y,z] = F[x^q - x, y, z]F[x, y^q - y, z]F[x, y, z^q - z]$$

is exactly $V(\mathcal{A})$. The polynomial $G \in I(V(\mathcal{A}))$ but it has monomial terms in single variable x,y,z hence does not belong to $\mathcal{A}$.

(5) *For finite fields $k = \mathbb{F}_q$ we have $\mathcal{A} \neq I(V(\mathcal{A}))$ even though both are radical ideals.*

(6) *Now if k is not a finite field then the class of polynomials functions agree with the class of polynomials. The alternative description of the ideal $\mathcal{A} = \{f : k^3 \longrightarrow k \mid f(\alpha,0,0) = f(0,\beta,0) = f(0,\gamma,0) = 0 \text{ for all } \alpha,\beta,\gamma \in k\}$. This is because if $f(\alpha,0,0) = 0$ for all $\alpha \in k$ then f does not have monomial terms only in x. Similarly f does not have monomial terms only in y or only in z. So $f \in \mathcal{A}$.*

(7) *For an infinite field $\mathcal{A} = I(V(\mathcal{A}))$.*

(8) *The ideal $\mathcal{A}$ is a homogeneous ideal. It does not have degree zero or degree one polynomials. Now $\mathcal{A}$ cannot be generated by two elements because if*

$$k[x,y,z] = V_0 \oplus V_1 \oplus V_2 \oplus \ldots$$

and

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \ldots$$

where $\mathcal{A}_i = \mathcal{A} \cap V_i$ then $dim_k(\mathcal{A}_0) = 0, dim_k(\mathcal{A}_1) = 0, dim_k(\mathcal{A}_2) = 3$. Hence $\mathcal{A}$ cannot be generated by 2-elements because $dim_k(\mathcal{A}_2) = 3$.

8. Problem 8

PROBLEM 8 (Geometry 8). *Don't assume k is algebraically closed for this problem. Let $\mathcal{A} = (x^2 + y^2 - 1, y - 1) = (x^2, y - 1)$. Describe $I(V(\mathcal{A}))\backslash\mathcal{A}$.*

SOLUTION 8. *For any field k we have $V(\mathcal{A}) = \{(0,1)\}$. So $I(V(\mathcal{A})) = (x, y-1)$.*

$$I(V(\mathcal{A}))\backslash\mathcal{A} = \{f(x,y) \in k[x,y] \text{ such that } x \mid f(x,1), x^2 \nmid f(x,1)\}.$$

We have $f(x,y) - f(x,1) = (y-1)h(x,y)$. First we observe that $f(x,y) \in I(V(\mathcal{A}))$ if and only if $x \mid f(x,1)$. If $f(x,y) = \alpha(x,y)x + \beta(x,y)(y-1) \in \mathcal{A}$ if and only if $\alpha(x,y)x + \beta(x,y)(y-1) = \gamma(x,y)x^2 + \delta(x,y)(y-1)$. So we get $f(x,1) = \alpha(x,1)x = \gamma(x,1)x^2$ which holds if and only if $x \mid \alpha(x,1)$ which holds if and only if $x^2 \mid f(x,1)$.

9. Problem 9

PROBLEM 9 (Geometry 9). *Let $X = V(x_1x_4 - x_2x_3) \subset \mathbb{A}^4_k$. Show that $A(X)$ is a UFD? Find a height 1 prime ideal which is not principal. Let $\phi : X \longrightarrow \mathbb{A}^1_k$ be a rational morphism defined as $\phi((x_1,x_2,x_3,x_4) \in X) = \frac{x_1}{x_2}$ then find the domain of ϕ.*

SOLUTION 9. *Consider the element* $t = \overline{x_1x_4} = \overline{x_2x_3}$. *This element as two different factorizations into irreducibles* $\overline{x_1}, \overline{x_2}, \overline{x_3}, \overline{x_4}$. *Let us prove the element* $\overline{x_1}$ *is irreducible. Suppose* $\overline{x_1} = \overline{f(x_1,x_2,x_3,x_4)g(x_1,x_2,x_3,x_4)}$ *then* $(x_1x_4 - x_2x_3) \mid (fg - x_1)$. *i.e.*

$$f(x_1,x_2,x_3,x_4)g(x_1,x_2,x_3,x_4) - x_1 = h(x_1,x_2,x_3,x_4)(x_1x_4 - x_2x_3).$$

Now substitute for x_4 *the term* $\frac{x_2x_3}{x_1}$. *We get that*

$$f(x_1,x_2,x_3,\frac{x_2x_3}{x_1})g(x_1,x_2,x_3,\frac{x_2x_3}{x_1}) - x_1 = 0.$$

Now let

$$f(x_1,x_2,x_3,\frac{x_2x_3}{x_1}) = \frac{F(x_1,x_2,x_3)}{x_1^m}, g(x_1,x_2,x_3,\frac{x_2x_3}{x_1}) = \frac{G(x_1,x_2,x_3)}{x_1^n}$$

Now we have in the UFD $k[x_1,x_2,x_3]$ $FG = x_1^{m+n+1}$. *Hence we get* $F(x_1,x_2,x_3) = \alpha x_1^r, G(x_1,x_2,x_3) = \frac{1}{\alpha}x_1^s$ *where* $r+s = m+n+1$ *with* $\alpha \in k^*$. *Now substituting above* $x_2 = 0, x_3 = 0$ *we get that*

$$f(x_1,x_2,x_3,\frac{x_2x_3}{x_1}) \overset{identically\ \ equal}{\equiv} f(x_1,0,0,0) = \alpha x_1^{r-m}$$

and

$$g(x_1,x_2,x_3,\frac{x_2x_3}{x_1}) \overset{identically\ \ equal}{\equiv} g(x_1,0,0,0) = \frac{1}{\alpha}x_1^{s-n}.$$

So we conclude the following.

- $r \geq m, s \geq n$ *as* $f(x_1,0,0,0), g(x_1,0,0,0)$ *are polynomials in* x_1 *with* $r+s = m+n+1$.
- *By factor theorem applied in the UFD* $k[x_1,x_2,x_3]_{\{1,x_1,x_1^2,\ldots,\}}[x_4]$

  $$f(x_1,x_2,x_3,x_4) - \alpha x_1^{r-m} = (x_4 - \frac{x_2x_3}{x_1})\frac{K(x_1,x_2,x_3,x_4)}{x_1^t}$$

  for some $K \in k[x_1,x_2,x_3]_{\{1,x_1,x_1^2,\ldots,\}}[x_4]$. *Now LHS is a polynomial and RHS has a denominator in* x_1. *Hence clearing denominators in RHS and using UFD property of the ring* $k[x_1,x_2,x_3,x_4]$ *we get*

  $$f(x_1,x_2,x_3,x_4) - \alpha x_1^{r-m} = (x_4x_1 - x_2x_3)K_1$$

  where K_1 *is a polynomial in the four variables* x_1,x_2,x_3,x_4. *Similarly*

  $$g(x_1,x_2,x_3,x_4) - \frac{1}{\alpha}x_1^{s-n} = (x_4x_1 - x_2x_3)K_2$$

  where K_2 *is a polynomial in the four variables* x_1,x_2,x_3,x_4.
  Hence the factorization reduces to

  $$\overline{x_1} = \alpha\overline{x_1}^{r-m}\frac{1}{\alpha}\overline{x_1}^{s-n}$$

  with $r \geq m, s \geq n$ *and* $r+s = m+n+1$. *This implies either* $r = m+1, n = s$ *or* $r = m, s = n+1$. *Hence we conclude that this factorization is a trivial factorization. So* $\overline{x_1}$ *is irreducible.*

Similarly $\overline{x_2}, \overline{x_3}, \overline{x_4}$ are also irreducibles.
The factorizations are clearly different because the set

$$\{\alpha_1\overline{x_1}, \alpha_2\overline{x_2}, \alpha_3\overline{x_3}, \alpha_4\overline{x_4}\}$$

with $\alpha_i \in k^* : i = 1, 2, 3, 4$ is a $4-$set i.e. it has four distinct elements of the ring $A(X)$ because the ideal $(x_1x_4 - x_2x_3)$ is a homogeneous ideal which has no degree 1 elements. This proves that $A(X)$ is not a UFD.
The prime ideal $(\overline{x_1}, \overline{x_2})$ is an ideal which is prime ideal of height one and it is not principal. Under the quotient map $k[x_1, x_2, x_3, x_4] \longrightarrow \frac{k[x_1,x_2,x_3,x_4]}{(x_1x_4-x_2x_3)}$, the ideal (x_1, x_2) is a prime in the polynomial ring containing the kernel produces a prime ideal in the quotient by the bijective correspondence of prime ideals which contain the kernel under the quotient map.
This is a prime ideal of height one because dimension of the ring goes down by one when we quotient by the principle prime ideal $(x_1x_4 - x_2x_3)$ which is an irreducible (non-unit, non-zerodivisor) using dimension theory. Hence the height of (x_1, x_2) which was 2 in the polynomial ring goes down by 1 in the quotient. So $(\overline{x_1}, \overline{x_2})$ is a height one prime ideal. This is clearly not principal as $\overline{x_1}, \overline{x_2}$ are two different irreducibles which do not differ by units.
The domain of the rational map $\phi : X \longrightarrow \mathbb{A}^1$ is given by

$$\{(x_1, x_2, x_3, x_4) \in X \mid \text{ either } x_2 \neq 0 \text{ or } x_4 \neq 0\} \subset X.$$

Let $(x_1, x_2, x_3, x_4) \in X$. Suppose $x_2 = 0$ then we have $x_1x_4 = 0$ so if $x_4 \neq 0$ then we get an alternative definition for $\phi(x_1, x_2, x_3, x_4) = \frac{x_3}{x_4}$ which gives an extension of the domain. Now if both $x_2 = 0 = x_4$ then we observe the following.
If $t = \frac{\overline{x_1}}{\overline{x_2}} \in Quotient - Field(A(X))$ and $\frac{\overline{r_1}}{\overline{r_2}}$ is any other representative for t then we have $r_1x_2 - r_2x_1 \in (x_1x_4 - x_2x_3)$. Substituting $x_2 = 0$ we get that

$$x_4 \mid r_2(x_1, 0, x_3, x_4)$$

Hence the domain of definition of ϕ cannot be extended beyond the above set in X. This proves the problem.

10. Problem 10

PROBLEM 10 (Geometry 10). *Let $\phi : \mathbb{A}^1 \longrightarrow \mathbb{A}^n$ be given as $t \longrightarrow (t, t^2, t^3, \ldots, t^n)$. Show that the image of ϕ is an affine variety and show that ϕ is an isomorphism onto its image.*

SOLUTION 10. *Clearly the inverse polynomial map is the first projection and the image is precisely the zero set of the equations $(x_i - x_1^i : 1 \leq i \leq n)$. This is irreducible because $\mathbb{A}^1$ is irreducible. This proves the problem.*

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